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ON A BOTTLENECK BIPARTITION CONJECTURE OF ERDŐS

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For a graph G, let $\gamma(U,V)=\max\{e(U),e(V)\}$ for a bipartition (U,V) of $V(G)$ with $U\cup V=V(G)$, $U\cap V=\emptyset$. Define $\gamma(G)=\min_{(U,V)}\{\gamma(U,V)\}$. Paul Erdős conjectures $\gamma(G)/e(G)\le 1/4+O\left(1/\sqrt{e(G)}\right)$. This paper verifies the conjecture and shows $\gamma(G)/e(G)\le 1/4\left(1+\sqrt{2/e(G)}\right)$.

This paper concerns simple graphs. The notation used is the following: For U, $V\subset V(G)$, $U\cap V=\emptyset$, $e(U)$ denotes the number of edges in the induced graph $G[U]$, $e(G)$ denotes the number of edges in G, and $e[U,V]=|\{uv|uv\in E(G), u\in U, v\in V\}|$. For a bipartition (U,V) of $V(G)$ with $U\cup V=V(G)$, $U\cap V=\emptyset$, let $\gamma(U,V)=\max\{e(U),e(V)\}$ and define $\gamma(G)=\min_{(U,V)}\{\gamma(U,V)\}$. The function $\gamma(G)$ was introduced by R. Entringer [2]. At the Sixth International Conference at Kalamazoo in 1988, Paul Erdős presented the conjecture that $\gamma(G)/e(G)\le 1/4+O\left(1/\sqrt{e(G)}\right)$. He recognized that the second-moment method is not going to give this result. Also, the computation of $\gamma(G)$ in NP-hard, as was proved by L. Clark, F. Shahroki, and L. A. Székely [1]. Therefore, one may not expect an algorithmic proof of Erdős' conjecture. This paper gives a non-constructive and non-probabilistic proof.

Theorem.

$$\frac{\gamma(G)}{e(G)}\le\frac{1}{4}\left(1+\sqrt{\frac{2}{e(G)}}\right).$$

A series of lemmas will give the proof of the theorem.

Given G, define $\Omega=\max_{(U,V)}\{e[U,V]\}$. Since $V(G)$ is finite, Ω is well-defined, it is known as the max cut of G. Define $S=\{(U,V)\mid e[U,V]=\Omega\}$, note $S\neq\emptyset$. For $v\in V(G)$, $H\subset V(G)$ defined $d_H(v)$ to be the number of vertices in H adjacent to v.

Lemma 1. *For any $(U,V)\in S$, $e[U,V]\ge 2\gamma(U,V)$.*

Proof. Let $e(U)=\max\{e(U),e(V)\}=\gamma(U,V)$. Since (U,V) defines a max cut, $d_V(x)\ge d_U(x)$, for all $x\in U$. Then, we have $e[U,V]=\sum_{x\in U} d_V(x)\ge\sum_{x\in U} d_U(x)=2e(U)=2\gamma(U,V)$. ∎

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Now, define $T = \{(U,V) \mid e[U,V] \geq 2\gamma(U,V)\}$. Since $S \subset T$, $T \neq \emptyset$. Let $\alpha = \min_T \gamma(U,V)$. Throughout, let (A,B) be a fixed partition on T realizing α, that is, $\gamma(A,B)=\alpha$. Let $e(A)=\max\{e(A),e(B)\}$ and define $\Delta=e(A)-e(B)$. Note if $\Delta=0$, then

$$\frac{e(A)}{e(G)} = \frac{e(A)}{2e(A)+e[A,b]} \leq \frac{e(A)}{4e(A)} = \frac{1}{4}.$$

The inequality, since $e[A,B] \geq 2\gamma(A,B) = 2e(A)$ for $(A,B) \in T$. So, assume $\Delta > 0$. Define $X \cup Y \subset A$ as follows: $X = \{x \in A \mid d_B(x) > \Delta/2\}$; $Y = \{y \in A \mid d_A(y) \neq 0$ and $d_B(y) \leq \Delta/2\}$. Note, $X \cap Y = \emptyset$. We then have $e(A)=e(X)+e[X,Y]+e(Y)$.

Lemma 2. *For all* $y \in Y$, $d_A(y) \leq d_B(y)$.

Proof. Assume there exists $y \in Y$ with $d_A(y) > d_B(y)$, then the partition $(A-y, B+y)$ contradicts the definition of (A,B), since with $e(B)=e(A)-\Delta$, we have $\gamma(A-y,B+y)=\max\{e(A)-d_A(y), e(B)+d_B(y)\}=\max\{e(A)-d_A(y), e(A)+d_B(y)-\Delta\} < e(A)=\alpha$ since $d_A(y)\neq 0$ and $d_B(y)\leq \Delta/2$. But then, $(A-y,B+y)$ is also in T since $e[A-y,B+y]=e[A,B]+(d_A(y)-d_B(y)) > e[A,B] \geq 2e(A) > 2\gamma(A-y,B+y)$. Consequently, we have $(A-y,B+y)\in T$ and $\gamma(A-y,B+y)<\alpha$, contradicting the definition of α. ∎

Lemma 3. *For all* $y \in Y$, $d_B(y) > 3d_A(y)$.

Proof. Assume there exists $y \in Y$ with $d_B(y) \leq 3d_A(y)$, then the partition $(A-y,B+y)$ contradicts the definition of (A,B). That is, $\gamma(A-y,B+y) = \max\{e(A)-d_A(y), e(B)+d_B(y)\} = \max\{e(A)-d_A(y), e(A)-\Delta+d_B(y)\} = e(A)-d_A(y) < e(A)=\alpha$, where $\max\{e(A)-d_A(y), e(A)-\Delta+d_B(y)\}=e(A)-d_A(y)$, since $d_A(y) \leq d_B(y) \leq \Delta/2$ by Lemma 2. But then, $(A-y,B+y)$ is also in T, since if $d_B(y) \leq 3d_A(y)$, $e[A-y,B+y] = e[A,B]-(d_B(y)-d_A(y)) \geq e[A,B]-2d_A(y) \geq 2\gamma(A,b)-2d_A(y) = 2\gamma(A-y,B+y)$. The last inequality holds because $e[A,B]\geq 2\gamma(A,B)$ for $(A,B)\in T$. Consequently, $(A-y,B+y)\in T$ and $\gamma(A-y,B+y)<\alpha$ contradicting the definition of α. ∎

With $e(A)=e(X)+e(Y)+e[X,Y]$, define ξ by $\xi \sum_{y\in Y} d_A(y) = e(Y)+e[X,Y]$. Then, $\xi=0$ or, $1/2 \leq \xi \leq 1$, where the extreme cases $\xi=1/2$, 1 indicate $e[X,Y]=0$, resp., $e(Y)=0$, and $\xi=0$ if and only if $Y=\emptyset$.

Lemma 4. *If* $e(X)=0$, *then* $e(A)/e(G)<1/4$.

Proof. If $Y=\emptyset$ and $e(X)=0$ then $e(A)/e(G)=0$, so assume $Y \neq \emptyset$; then for $e(X)=0$, $e(A)=e(Y)+e[X,Y]=\xi \sum_{y\in Y} d_A(y)$. Observe that $e[A,B] = \sum_{y\in A} d_B(y) \geq \sum_{y\in X\cup Y} d_B(y) \geq \sum_{y\in Y} d_B(y) > 3\sum_{y\in Y} d_A(y)$ follows from Lemma 3.

Hence

$$\frac{e(A)}{e(G)} = \frac{e(A)}{e(A)+e[A,B]+e(B)} = \frac{\xi \sum_{y\in Y} d_A(y)}{\xi \sum_{y\in Y} d_A(y) + e[A,B] + e(B)}$$
$$\leq \frac{\xi \sum_{y\in Y} d_A(y)}{\xi \sum_{y\in Y} d_A(y) + e[A,B]} < \frac{\xi \sum_{y\in Y} d_A(y)}{\xi \sum_{y\in Y} d_A(y) + 3\sum_{y\in Y} d_A(y)} \leq \frac{1}{4}$$

follows since $\xi \le 1$. ∎

For $e(X) \neq 0$, let $k = |X|$. Then, $e(X) = ck(k-1)/2$ for some $c \le 1$. We have

$$\frac{e(A)}{e(G)} = \frac{e(A)}{2e(A) + e[A,B] - \Delta} = \frac{e(X) + \xi \sum_{y \in Y} d_A(y)}{2\left(e(X) + \xi \sum_{y \in Y} d_A(y)\right) + e[A,B] - \Delta}.$$

Case 1: $\Delta \le \lfloor 2c(k-1) \rfloor$.

We have $e[A,B] \ge 2e(A)$, since $(A,B) \in T$. Consequently,

$$\frac{e(A)}{e(G)} \le \frac{e(A)}{4e(A) - \Delta} = \frac{e(X) + \xi \sum_{y \in Y} d_A(y)}{4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y)}.$$

Define $\eta \ge 0$, by $e(G) = 4e(X) - \Delta + 4\xi \sum\limits_{y \in Y} d_A(y) + \eta$. We first show

$$\frac{e(X)}{4e(X) - \Delta} \le \frac{1}{4}\left(1 + \sqrt{\frac{2}{4e(X) - \Delta}}\right).$$

Let $\lambda = 4e(X) - 2c(k-1) = 2c(k-1)^2$, then $4e(X) - \Delta = \lambda + \varepsilon$ for some $\varepsilon \ge 0$, since $\Delta \le \lfloor 2c(k-1) \rfloor$. We have

$$\frac{e(X)}{\lambda} = \frac{ck(k-1)/2}{2c(k-1)^2} = \frac{k}{4(k-1)} \le \frac{1}{4}\left(1 + \sqrt{\frac{2}{\lambda}}\right) = \frac{1}{4}\left(1 + \sqrt{\frac{2}{2c(k-1)^2}}\right),$$

since with $c \le 1$, $\sqrt{(1/c)} \ge 1$. Consequently,

$$e(X) \le \frac{1}{4}(\lambda + \sqrt{2\lambda}) \le \frac{1}{4}(\lambda + \varepsilon + \sqrt{2(\lambda + \varepsilon)}) = \frac{1}{4}\left(4e(X) - \Delta + \sqrt{2(4e(X) - \Delta)}\right).$$

Hence

$$\frac{e(X)}{4e(X) - \Delta} \le \frac{1}{4}\left(1 + \sqrt{\frac{2}{4e(X) - \Delta}}\right).$$

Consequently,

$$\begin{aligned}
e(A) &= e(X) + \xi \sum_{y \in Y} d_A(y) \\
&\le \frac{1}{4}\left(4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y) + \sqrt{2\left(4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y)\right)}\right) \\
&\le \frac{1}{4}\left(\left(4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y) + \eta\right) + \sqrt{2\left(4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y) + \eta\right)}\right) \\
&= \frac{1}{4}(e(G) + \sqrt{2e(G)}).
\end{aligned}$$

The last inequality holds since $\eta \geq 0$. Hence $e(A)/e(G) \leq (1/4) \cdot \left(1+\sqrt{2/e(G)}\right)$.

Case 2: $\Delta > \lfloor 2c(k-1) \rfloor$.

We have

$$\frac{e(A)}{e(G)} = \frac{e(A)}{2e(A)+e[A,B]-\Delta} = \frac{e(X)+\xi\sum_{y\in Y} d_A(y)}{2e(X)+e[X,B]-\Delta+2\xi\sum_{y\in Y} d_A(y)+e[A\setminus X,B]}.$$

We first show

$$\frac{e(X)}{2e(X)+e[A,B]-\Delta} \leq \frac{1}{4}\left(1+\sqrt{\frac{2}{2e(X)+e[X,B]-\Delta}}\right),$$

which will be sufficient since

$$(1) \qquad \xi \sum_{y\in Y} d_A(y) \leq \frac{1}{4}\left(2\xi \sum_{y\in Y} d_A(y) + e[A\setminus X, B]\right).$$

The inequality (1) is trivial if $\xi = 0$, so assume $1/2 \leq \xi \leq 1$. Then with $Y \subset A\setminus X$ and using Lemma 3 gives $e([A\setminus X, B] \geq e[Y,B] > 3\sum_{y\in Y} d_A(y)$. Hence,

$$\frac{\xi\sum_{y\in Y} d_A(y)}{2\xi\sum_{y\in Y} d_A(y)+e[A\setminus X,B]} \leq \frac{\xi\sum_{y\in Y} d_A(y)}{2\xi\sum_{y\in Y} d_A(y)+3\sum_{y\in Y} d_A(y)} \leq \frac{1}{5}.$$

The last inequality holds, since $\xi \leq 1$. This establishes (1). We have

$$\frac{e(X)}{2e(X)+e[X,B]-\Delta} < \frac{e(X)}{2e(X)+k\frac{\Delta}{2}-\Delta} = \frac{e(X)}{2e(X)+\frac{\Delta}{2}(k-2)}.$$

The inequality follows from the definition of X, i.e., $e[X,B] > |X|(\Delta/2) = (k\Delta/2)$. Let $\lambda = 2e(X) + (2c(k-1)/2)(k-2) = 2c(k-1)^2$. Then, with $\Delta \in \mathbb{N}$, we have $2e(X)+e[X,B]-\Delta = \lambda+\varepsilon$ for some $\varepsilon > 0$. We have

$$\frac{e(X)}{\lambda} = \frac{ck(k-1)/2}{2c(k-1)^2} = \frac{k}{4(k-1)} \leq \frac{1}{4}\left(1+\sqrt{\frac{2}{\lambda}}\right).$$

The last inequality was shown in Case 1. Consequently,

$$e(X) \leq \frac{1}{4}(\lambda+\sqrt{2\lambda}) < \frac{1}{4}\left(\frac{\lambda+\varepsilon}{4}+\sqrt{2(\lambda+\varepsilon)}\right)$$
$$= \frac{1}{4}\left(2e(X)+e[X,B]-\Delta+\sqrt{2(2e(X)+e[X,B]-\Delta)}\right).$$

Hence,

$$\frac{e(X)}{2e(X)+e[X,B]-\Delta} \leq \frac{1}{4}\left(1+\sqrt{\frac{2}{2e(X)+e[X,B]-\Delta}}\right).$$

Consequently,

$$e(A) = e(X) + \xi \sum_{y \in Y} d_A(y)$$

$$\leq \frac{1}{4}\left[\left(2e(X) + e[X,B] - \Delta + 2\xi \sum_{y\in Y} d_A(y) + e[A \setminus X, B]\right) + \left(\sqrt{2\left(2e(X) + e[X,B] - \Delta + 2\xi \sum_{y\in Y} d_A(y) + e[A \setminus X, B]\right)}\right)\right]$$

$$= \frac{1}{4}\left(e(G) + \sqrt{2e(G)}\right).$$

The last inequality holds from (1). Hence $e(A)/e(G) \leq (1/4)\cdot\left(1+\sqrt{2/e(G)}\right)$.

We conclude with the conjecture that the sharp bound for $\gamma(G)/e(G)$ is $(1/4)\cdot(1+1/\sqrt{2e(G)})$. The author has shown this upper bound for all r-regular graphs G. This bound is best possible, since for n odd, $\gamma(K_n)/e(K_n) \sim (1/4)\cdot(1+1/\sqrt{2e(K_n)})$.

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